A synthetical integrable two-component model with peakon solutions

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Abstract

In this paper, we propose a synthetical integrable two-component model with peakon solutions. The two-component model is proven integrable through its Lax pair and infinitely many conservation laws. Moreover, we investigate the bi-Hamiltonian structure and the interaction of multi-peakons from the two-component model. In the paper, we also study some reductions from the two-component model and obtain very interesting solutions, such as new type of N-peakon solutions, N-weak-kink solutions, weak hat-shape soliton solutions, periodic-peakon solutions, and rational peakon solutions. In particular, we first time propose N-peakon solutions which are not in the traveling wave type, N-weak-kink solutions, rational peakon solutions, and weak hat-shape soliton solutions.

Keywords: Integrable system, Peakon, Kink, Lax pair.
PACS: 02.30.Ik, 04.20.Jb.

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1 Introduction

In recent years, the Camassa-Holm (CH) equation [1]

$$m_t + 2mu_x + m_x u = 0, \quad m = u - u_{xx} + k, \tag{1}$$

(k is an arbitrary constant) derived by Camassa and Holm [1] as a shallow water wave model, has caught much attention and various studies. The CH equation admits Lax representation [1], bi-Hamiltonian structures [2],[3], and is integrable by the inverse scattering transformation [7]. Also it possesses multi-peakon solutions [4] and algebro-geometric solutions [5, 6]. The most interesting feature of the CH equation is: admitting peaked soliton (peakon) solutions in the case of $k = 0$ [1, 4]. A peakon is a weak solution in some Sobolev space with corner at its crest. The stability and interaction of peakons were discussed in several references [8]-[13].

In addition to the CH equation, other integrable models with peakon solutions have been found, such as the Degasperis-Procesi (DP) equation [14]

$$m_t + 3mu_x + m_x u = 0, \quad m = u - u_{xx}, \tag{2}$$

the cubic nonlinear peakon equation or modified CH equation [3, 17, 18, 19]

$$m_t = \left[m(u^2 - u_x^2)\right]_x, \quad m = u - u_{xx}, \tag{3}$$

the Novikov's cubic nonlinear equation [20]

$$m_t = u^2 m_x + 3uu_x m, \quad m = u - u_{xx}, \tag{4}$$

and a generalized CH equation with both quadratic and cubic nonlinearity [21]

$$m_t = \frac{1}{2}k_1\left[m(u^2 - u_x^2)\right]_x + \frac{1}{2}k_2(2mu_x + m_x u), \quad m = u - u_{xx}, \tag{5}$$

where k_1 and k_2 are two arbitrary constants.

The interesting characteristics of the CH equation stimulated more people to study peakon, cuspon, and weak kink solutions to enrich the theory of soliton and integrable systems. One of important tasks is to seek for integrable multi-component peakon systems with amazing integrable properties and applications. For example, the integrable two-component CH equation are proposed in [3, 22, 23, 24]. The integrable two-component forms of the cubic peakon equation (3) are presented in [25, 26].

In this paper, we propose a more generalized version of the integrable two-component peakon system as follows:

$$\begin{cases} m_t = F + F_x - \frac{1}{2}m\left(uv - u_x v_x + uv_x - u_x v\right), \\ n_t = -G + G_x + \frac{1}{2}n\left(uv - u_x v_x + uv_x - u_x v\right), \\ m = u - u_{xx}, \\ n = v - v_{xx}, \end{cases} \tag{6}$$

where F and G are two arbitrary functions of u, v and their derivatives satisfying

$$mG = nF. \tag{7}$$

As

$$v = 2, \quad G = 2u, \quad F = mu,$$

(6) is reduced to the CH equation (1). As

$$v = 2u, \quad F = m(u^2 - u_x^2), \quad G = 2m(u^2 - u_x^2),$$

(6) is reduced to the cubic peakon equation (3). As

$$v = k_1 u + k_2, \quad F = \frac{1}{2}k_1 m(u^2 - u_x^2) + \frac{1}{2}k_2 mu, \quad G = \frac{1}{2}k_1^2 m(u^2 - u_x^2) + \frac{1}{2}k_1 k_2(mu + u^2 - u_x^2) + \frac{1}{2}k_2^2 u,$$

(6) is reduced to the generalized CH equation (5). Thus, equation (6) is a kind of two-component generalizations of equation (1), (3) and (5). We show that the two-component model (6) is integrable in the sense of Lax pair and infinitely many conservation laws. By choosing $G = nH$ and $F = mH$, where H is an arbitrary polynomial in u, v and their derivatives, we may obtain quite a large number of integrable two-component systems with peakon solutions. As examples, the bi-Hamiltonian structures and the peakon interactions of some integrable two-component systems are discussed in detail. In the paper, we also study some reductions from the two-component model and obtain very interesting solutions, such as new type of N-peakon solutions, N-weak-kink solutions, weak hat-shape soliton solutions, periodic-peakon solutions, and rational peakon solutions. In particular, we first time propose new type of N-peakon solutions which are not in the traveling wave type, N-weak-kink solutions, weak hat-shape soliton solutions, and rational peakon solutions whose feature looks very like the so-called rogue wave [27, 28].

2 Lax pair and conservation laws

Let us consider a pair of 2×2 matrix spectral problems of the following type

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad U = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -\lambda n & 1 \end{pmatrix}, \tag{8}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = V \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad V = -\frac{1}{2}\begin{pmatrix} \lambda^{-2} + E & -\lambda^{-1}(u - u_x) - \lambda F \\ \lambda^{-1}(v + v_x) + \lambda G & -\lambda^{-2} - E \end{pmatrix}, \tag{9}$$

where $m = u - u_{xx}$, $n = v - v_{xx}$, $E = \frac{1}{2}(uv - u_x v_x + uv_x - u_x v)$, F and G are arbitrary polynomials in u, v and their derivatives satisfying the constraint (7). By direct calculations, we find the the compatibility condition of (8) and (9)

$$U_t - V_x + [U, V] = 0 \tag{10}$$

yields nothing but equation (6). Thus, (8) and (9) compose of a Lax pair of equation (6).

Next, let us construct the conservation laws for equation (6). Let $\omega = \frac{\phi_2}{\phi_1}$, then ω satisfies the following Riccati equation

$$\omega_x = -\frac{1}{2}\lambda m\omega^2 + \omega - \frac{1}{2}\lambda n. \tag{11}$$

Based on (8) and (9), we obtain

$$(\ln\phi_1)_x = -\frac{1}{2} + \frac{1}{2}\lambda m\omega, \quad (\ln\phi_1)_t = -\frac{1}{2}\left[\lambda^{-2} + E - \lambda^{-1}(u-u_x)\omega - \lambda F\omega\right], \tag{12}$$

which generates the following conservation law of equation (6):

$$\rho_t = A_x, \tag{13}$$

where

$$\begin{aligned} &\rho = m\omega, \\ &A = -\lambda^{-1}E + \lambda^{-2}(u-u_x)\omega + F\omega. \end{aligned} \tag{14}$$

ρ and A are usually called a conserved density and an associated flux, respectively. To derive the explicit forms of conservation densities, we expand ω in terms of negative powers of λ as below:

$$\omega = \sum_{j=0}^{\infty}\omega_j\lambda^{-j}. \tag{15}$$

Substituting (15) into (11) and equating the coefficients of powers of λ, we arrive at

$$\begin{aligned} \omega_0 &= \sqrt{-\frac{n}{m}}, \qquad \omega_1 = \frac{mn_x - m_x n - 2mn}{2m^2 n}, \\ \omega_{j+1} &= \frac{1}{m\omega_0}\left[\omega_j - \omega_{j,x} - \frac{1}{2}m\sum_{i+k=j+1,\ i,k\geq 1}\omega_i\omega_k\right], \quad j \geq 1. \end{aligned} \tag{16}$$

Inserting (15) and (16) into (14), we finally get the following infinitely many conserved densities and the associated fluxes

$$\begin{aligned} &\rho_0 = \sqrt{-mn}, \qquad A_0 = F\sqrt{-\frac{n}{m}}, \\ &\rho_1 = \frac{mn_x - m_x n - 2mn}{2mn}, \qquad A_1 = -E + \frac{(mn_x - m_x n - 2mn)F}{2m^2 n}, \\ &\rho_j = m\omega_j, \qquad A_j = (u-u_x)\omega_{j-2} + F\omega_j, \quad j \geq 2, \end{aligned} \tag{17}$$

where ω_j is given by (16).

3 Integrable two-component peakon systems

Guided by (7), we take F and G as

$$F = mH, \quad G = nH, \tag{18}$$

where H is an arbitrary polynomial in u, v and their derivatives. Then (6) becomes

$$\begin{cases} m_t = (mH)_x + \frac{1}{2}m\left(2H - uv + u_x v_x - u v_x + u_x v\right), \\ n_t = (mH)_x - \frac{1}{2}n\left(2H - uv + u_x v_x - u v_x + u_x v\right), \\ m = u - u_{xx}, \quad n = v - v_{xx}, \end{cases} \tag{19}$$

which admits the Lax pair

$$\begin{aligned} U &= \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -\lambda n & 1 \end{pmatrix}, \\ V &= -\frac{1}{2}\begin{pmatrix} \lambda^{-2} + \frac{1}{2}\left(uv - u_x v_x + u v_x - u_x v\right) & -\lambda^{-1}(u - u_x) - \lambda m H \\ \lambda^{-1}(v + v_x) + \lambda n H & -\lambda^{-2} - \frac{1}{2}\left(uv - u_x v_x + u v_x - u_x v\right) \end{pmatrix}. \end{aligned} \tag{20}$$

By choosing suitable H, we may obtain quite a lot of integrable systems possessing peakon solutions. Here we only discuss the following examples.

Example 3.1. A new integrable system with new type of peakon solutions

Taking $H = 0$ in equation (19) gives rise to the following integrable two-component model

$$\begin{cases} m_t = -\frac{1}{2}m\left(uv - u_x v_x + u v_x - u_x v\right), \\ n_t = \frac{1}{2}n\left(uv - u_x v_x + u v_x - u_x v\right), \\ m = u - u_{xx}, \quad n = v - v_{xx}. \end{cases} \tag{21}$$

This model can be rewritten as the following bi-Hamiltonian form

$$(m_t,\ n_t)^T = J\left(\frac{\delta H_2}{\delta m},\ \frac{\delta H_2}{\delta n}\right)^T = K\left(\frac{\delta H_1}{\delta m},\ \frac{\delta H_1}{\delta n}\right)^T, \tag{22}$$

where

$$J = \begin{pmatrix} 0 & -\partial - 1 \\ -\partial + 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} m\partial^{-1}m & -m\partial^{-1}n \\ -n\partial^{-1}m & n\partial^{-1}n \end{pmatrix}, \tag{23}$$

$$H_1 = \frac{1}{2}\int_{-\infty}^{+\infty}(u_x - u_{xx})n\,dx, \quad H_2 = \frac{1}{4}\int_{-\infty}^{+\infty}(u - u_x)^2(v + v_x)n\,dx. \tag{24}$$

Let us assume that (21) has the following one-peakon solution

$$u = p_1(t)e^{-|x - q_1(t)|}, \quad v = r_1(t)e^{-|x - q_1(t)|}, \tag{25}$$

where $p_1(t)$, $r_1(t)$ and $q_1(t)$ are functions of t needed to be determined. Substituting (25) into (21) and integrating against the test function with support around the peak, we obtain

$$p_{1,t} = -\frac{1}{3}p_1^2 r_1, \qquad r_{1,t} = \frac{1}{3}p_1 r_1^2, \qquad q_{1,t} = 0, \tag{26}$$

which yields

$$p_1(t) = A_2 e^{-\frac{1}{3}A_1 t}, \qquad r_1(t) = \frac{A_1}{A_2} e^{\frac{1}{3}A_1 t}, \qquad q_1(t) = A_3, \tag{27}$$

where A_1, A_2, and A_3 are integration constants. Thus, we obtain the peakon solutions as follows

$$u(x,t) = A_2 e^{-\frac{1}{3}A_1 t} e^{-|x-A_3|}, \qquad v(x,t) = \frac{A_1}{A_2} e^{\frac{1}{3}A_1 t} e^{-|x-A_3|}. \tag{28}$$

This pair of single-peakon solutions is not presented in the traveling wave type, because the peakon position $q_1(t) = A_3$ is stationary. As existed in the literature, within our knowledge, all the current integrable peakon models admit the single-peakon solutions in traveling wave type, such as CH equation [1], DP equation [15, 16], the cubic nonlinear models [19, 20, 21], and their two-component extensions in [23, 24, 26]. So, we find a new integrable peakon system (21) whose peakon solution is not in traveling wave type. See Figure 1 for the profile of the new single-peakon solution.

Let us suppose the N-peakon solution in the form of

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \quad v(x,t) = \sum_{j=1}^{N} r_j(t) e^{-|x-q_j(t)|}. \tag{29}$$

Substituting (29) into (21) and integrating against test functions, we obtain the N-peakon dynamic system of (21):

$$\begin{cases} q_{j,t} =0, \\ p_{j,t} =\frac{1}{6}p_j^2 r_j \\ \qquad + \frac{1}{2}p_j \sum\limits_{i,k=1}^{N} p_i r_k \left(sgn(q_j-q_i)sgn(q_j-q_k) - sgn(q_j-q_i) + sgn(q_j-q_k) - 1\right) e^{-|q_j-q_i|-|q_j-q_k|}, \\ r_{j,t} =-\frac{1}{6}p_j r_j^2 \\ \qquad - \frac{1}{2}r_j \sum\limits_{i,k=1}^{N} p_i r_k \left(sgn(q_j-q_i)sgn(q_j-q_k) - sgn(q_j-q_i) + sgn(q_j-q_k) - 1\right) e^{-|q_j-q_i|-|q_j-q_k|}. \end{cases} \tag{30}$$

In the above formula, $q_{j,t} = 0$ implies that the peak position does not change along with the time t.

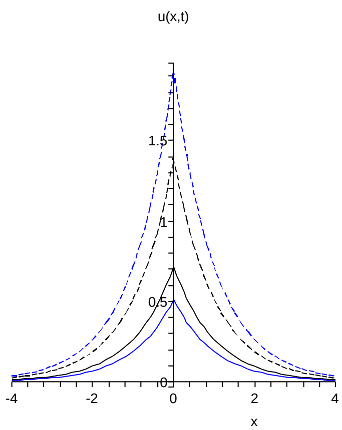

Figure 1: The single-peakon solution given by (28) with $A_1 = A_2 = 1$ and $A_3 = 0$. Solid line: $u(x,t)$; Dashed line: $v(x,t)$; Black: $t = 1$; Blue: $t = 2$.

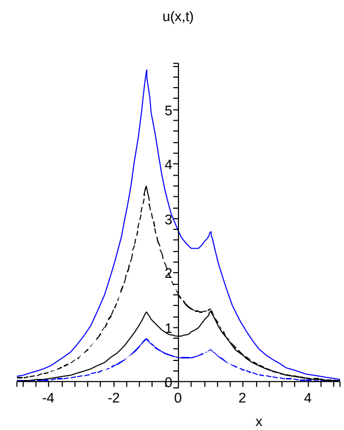

Figure 2: The two-peakon solution given by (32). Solid line: $u(x,t)$; Dashed line: $v(x,t)$; Black: $t = 0$; Blue: $t = -1$.

For $N = 2$, solving (30) leads to

$$\begin{cases} q_1(t) = A_4, \quad q_2(t) = A_5, \\ p_1(t) = A_6 e^{-\frac{1}{3}A_1 t - \frac{e^{-|A_4-A_5|}}{2}\left[\frac{3A_1(1+sgn(A_4-A_5))}{(A_1-A_2)A_3}e^{\frac{1}{3}(A_1-A_2)t} - \frac{3A_2A_3(1-sgn(A_4-A_5))}{A_1-A_2}e^{-\frac{1}{3}(A_1-A_2)t}\right]}, \\ p_2(t) = \frac{p_1}{A_3}e^{\frac{1}{3}(A_1-A_2)t}, \\ r_1(t) = \frac{A_1}{p_1}, \quad r_2(t) = \frac{A_2}{p_2}, \end{cases} \tag{31}$$

where $A_1,\ A_2,\ \cdots\ A_6$ are integration constants. If $A_4 = A_5$, it is reduced to the one-peakon solution. If $A_4 \neq A_5$, this two-peakon solution will never collide because $q_1(t) \neq q_2(t)$ for any t. In particular, for $A_1 = A_3 = A_4 = -A_5 = A_6 = 1$ and $A_2 = 4$, the two-peakon becomes

$$\begin{cases} u(x,t) = e^{-\frac{1}{3}t + e^{-t-2}}e^{-|x-1|} + e^{-\frac{4}{3}t + e^{-t-2}}e^{-|x+1|}, \\ v(x,t) = e^{\frac{1}{3}t - e^{-t-2}}e^{-|x-1|} + 4e^{\frac{4}{3}t - e^{-t-2}}e^{-|x+1|}. \end{cases} \tag{32}$$

See Figure 2 for the profile of the two-peakon dynamics for the potentials $u(x,t)$ and $v(x,t)$.

Example 3.2. The integrable two-component system proposed in [26]

Choosing $H = \frac{1}{2}(uv - u_x v_x)$, we obtain

$$\begin{cases} m_t = \frac{1}{2}[m(uv - u_x v_x)]_x - \frac{1}{2}m(uv_x - u_x v), \\ n_t = \frac{1}{2}[n(uv - u_x v_x)]_x + \frac{1}{2}n(uv_x - u_x v), \\ m = u - u_{xx}, \quad n = v - v_{xx}, \end{cases} \tag{33}$$

which is exactly the system we derived in [26]. This system possesses the bi-Hamiltonian form

$$(m_t,\ n_t)^T = J\left(\frac{\delta H_1}{\delta m},\ \frac{\delta H_1}{\delta n}\right)^T = K\left(\frac{\delta H_2}{\delta m},\ \frac{\delta H_2}{\delta n}\right)^T, \tag{34}$$

where

$$J = \begin{pmatrix} 0 & \partial^2 - 1 \\ 1 - \partial^2 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} \partial m \partial^{-1} m \partial - m \partial^{-1} m & \partial m \partial^{-1} n \partial + m \partial^{-1} n \\ \partial n \partial^{-1} m \partial + n \partial^{-1} m & \partial n \partial^{-1} n \partial - n \partial^{-1} n \end{pmatrix}, \tag{35}$$

$$H_1 = \frac{1}{2}\int_{-\infty}^{+\infty} (uv + u_x v_x) dx, \quad H_2 = \frac{1}{4}\int_{-\infty}^{+\infty} (u^2 v_x + u_x^2 v_x - 2uu_x v) n dx. \tag{36}$$

In [26], we have derived the one-peakon of (33)

$$u(x,t) = c_1 e^{-|x + \frac{1}{3}c_1 c_2 t|}, \qquad v(x,t) = c_2 e^{-|x + \frac{1}{3}c_1 c_2 t|}, \tag{37}$$

where c_1 and c_2 are two arbitrary integration constants. We also investigated the N-peakon dynamical system. In particular, the two-peakon solution was given explicitly and the collisions are discussed (for details, see [26]).

Example 3.3. A new integrable two-component peakon system with the same bi-Hamiltonian operators as (33) but different Hamiltonian functions

Taking $H = \frac{1}{2}(uv_x - u_x v)$, we arrive at

$$\begin{cases} m_t = \frac{1}{2}\left(m(uv_x - u_x v)\right)_x - \frac{1}{2} m\left(uv - u_x v_x\right), \\ n_t = \frac{1}{2}\left(n(uv_x - u_x v)\right)_x + \frac{1}{2} n\left(uv - u_x v_x\right), \\ m = u - u_{xx}, \quad n = v - v_{xx}. \end{cases} \tag{38}$$

This system can be rewritten as the following bi-Hamiltonian form

$$(m_t,\ n_t)^T = J\left(\frac{\delta H_2}{\delta m},\ \frac{\delta H_2}{\delta n}\right)^T = K\left(\frac{\delta H_1}{\delta m},\ \frac{\delta H_1}{\delta n}\right)^T, \tag{39}$$

where J, K are given by (35), and

$$H_1 = \frac{1}{2}\int_{-\infty}^{+\infty} (uv_x + u_x v_{xx}) dx, \quad H_2 = \frac{1}{4}\int_{-\infty}^{+\infty} (u^2 v + u_x^2 v - 2uu_x v_x) n dx. \tag{40}$$

From (34) and (39), we know that equation (33) and (38) share the same bi-Hamiltonian operators but with different Hamiltonian functions.

By direct calculations, we find that the one-peakon solution of (38) takes the form as

$$u(x,t) = c_2 e^{-\frac{1}{3}c_1 t} e^{-|x - c_3|}, \qquad v(x,t) = \frac{c_1}{c_2} e^{\frac{1}{3}c_1 t} e^{-|x - c_3|}, \tag{41}$$

where c_1, c_2 and c_3 are three integration constants. In general, we obtain the N-peakon dynam-

ical system of (38):

$$\begin{cases} p_{j,t} = \dfrac{1}{6}p_j^2 r_j + \dfrac{1}{2}p_j \displaystyle\sum_{i,k=1}^{N} p_i r_k \left(sgn(q_j - q_i) sgn(q_j - q_k) - 1\right) e^{-|q_j - q_i| - |q_j - q_k|}, \\ r_{j,t} = -\dfrac{1}{6}p_j r_j^2 - \dfrac{1}{2}r_j \displaystyle\sum_{i,k=1}^{N} p_i r_k \left(sgn(q_j - q_i) sgn(q_j - q_k) - 1\right) e^{-|q_j - q_i| - |q_j - q_k|}, \\ q_{j,t} = \dfrac{1}{2} \displaystyle\sum_{i,k=1}^{N} p_i r_k \left(sgn(q_j - q_k) - sgn(q_j - q_i)\right) e^{-|q_j - q_i| - |q_j - q_k|}. \end{cases} \tag{42}$$

For $N = 2$, the two-peakon dynamical system reads as

$$\begin{cases} p_{1,t} = -\frac{1}{3}p_1^2 r_1 - \frac{1}{2}p_1 \left(p_1 r_2 + p_2 r_1\right) e^{-|q_1 - q_2|}, \\ p_{2,t} = -\frac{1}{3}p_2^2 r_2 - \frac{1}{2}p_2 \left(p_1 r_2 + p_2 r_1\right) e^{-|q_1 - q_2|}, \\ r_{1,t} = \frac{1}{3}p_1 r_1^2 + \frac{1}{2}r_1 \left(p_1 r_2 + p_2 r_1\right) e^{-|q_1 - q_2|}, \\ r_{2,t} = \frac{1}{3}p_2 r_2^2 + \frac{1}{2}r_2 \left(p_1 r_2 + p_2 r_1\right) e^{-|q_1 - q_2|}, \\ q_{1,t} = \frac{1}{2}\left(p_1 r_2 - p_2 r_1\right) sgn(q_1 - q_2) e^{-|q_1 - q_2|}, \\ q_{2,t} = q_{1,t}. \end{cases} \tag{43}$$

From the first four equations of (43), we may conclude $p_1(t)r_1(t) = A_1$ and $p_2(t)r_2(t) = A_2$ where A_1 and A_2 are two integration constants. From the last two equations of (43), we know $q_2(t) = q_1(t) - B_1$ where B_1 is a nonzero constant, which indicates that the two-peakon will never collide. For $A_1 = A_2$, we have

$$\begin{cases} p_1(t) = De^{\left[-\frac{1}{3}A_1 - \frac{1}{2}\left(A_1 C_1 + \frac{A_1}{C_1}\right)e^{-|B_1|}\right]t}, \\ p_2(t) = \frac{p_1(t)}{C_1}, \qquad r_1(t) = \frac{A_1}{p_1(t)}, \qquad r_2 = \frac{A_1 C_1}{p_1(t)}, \\ q_1(t) = \frac{1}{2}\left[(A_1 C_1 - \frac{A_1}{C_1}) sgn(B_1) e^{-|B_1|}\right]t + \frac{B_1}{2}, \\ q_2(t) = q_1(t) - B_1, \end{cases} \tag{44}$$

where B_1, C_1, and D are three integration constants. For example, choosing $C_1 = D = 1$, $B_1 = 2$, $A_1 = 3$, we have $p_2(t) = p_1(t) = e^{-(3e^{-2}+1)t}$, $r_2(t) = r_1(t) = 3e^{(3e^{-2}+1)t}$. Thus, the two-peakon solution accordingly reads as

$$\begin{cases} u(x,t) = e^{-(3e^{-2}+1)t}\left(e^{-|x-1|} + e^{-|x+1|}\right), \\ v(x,t) = 3e^{(3e^{-2}+1)t}\left(e^{-|x-1|} + e^{-|x+1|}\right), \end{cases} \tag{45}$$

which are apparently M-shape peakon solutions with two peaks (see Figure 3 for details). If choosing $C_1 = B_1 = 2$, $D = 1$, $A_1 = 3$, then we have the following two-peakon solution

$$\begin{cases} u(x,t) = \frac{1}{2}e^{-(\frac{15}{4}e^{-2}+1)t}\left(2e^{-|x-\frac{9}{4}e^{-2}t-1|} + e^{-|x-\frac{9}{4}e^{-2}t+1|}\right), \\ v(x,t) = 3e^{(\frac{15}{4}e^{-2}+1)t}\left(e^{-|x-\frac{9}{4}e^{-2}t-1|} + 2e^{-|x-\frac{9}{4}e^{-2}t+1|}\right). \end{cases} \tag{46}$$

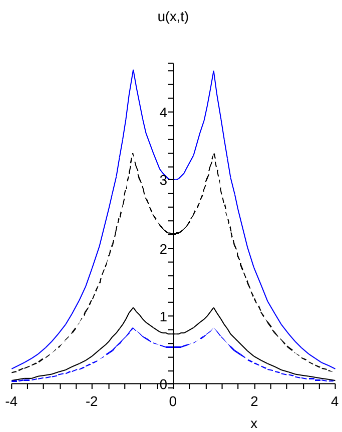

Figure 3: The M-shape peakon solution given by (45). Solid line: $u(x,t)$; Dashed line: $v(x,t)$; Black: $t = 0$; Blue: $t = -1$.

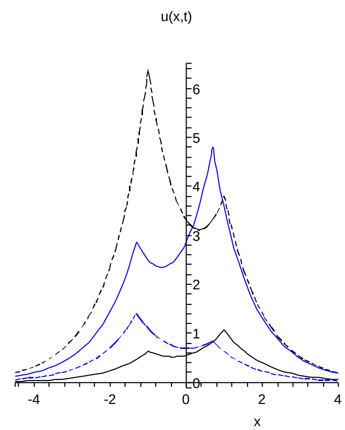

Figure 4: The two-peakon solution given by (46). Solid line: $u(x,t)$; Dashed line: $v(x,t)$; Black: $t = 0$; Blue: $t = -1$.

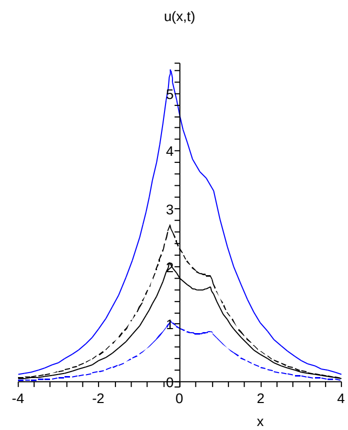

Figure 5: The two-peakon solution determined by (48). Solid line: $u(x,t)$; Dashed line: $v(x,t)$; Black: $t = -0.5$; Blue: $t = -1$.

Figure 4 shows the profile of this two-peakon solution.

For $A_1 \neq A_2$, we obtain the following solution of (43):

$$\begin{cases} p_1(t) = B_3 e^{-\frac{1}{3}A_1 t - \frac{3e^{-|B_1|}}{2(A_1-A_2)}\left(\frac{A_1}{B_2}e^{\frac{1}{3}(A_1-A_2)t} - A_2B_2e^{-\frac{1}{3}(A_1-A_2)t}\right)}, \\ p_2(t) = \frac{p_1}{B_2}e^{\frac{1}{3}(A_1-A_2)t}, \\ r_1(t) = \frac{A_1}{p_1}, \qquad r_2 = \frac{A_2}{p_2} \\ q_1(t) = -\frac{3sgn(B_1)e^{-|B_1|}}{2(A_1-A_2)}\left[A_2B_2e^{-\frac{1}{3}(A_1-A_2)t} + \frac{A_1}{B_2}e^{\frac{1}{3}(A_1-A_2)t}\right] + B_4, \\ q_2(t) = q_1 - B_1, \end{cases} \tag{47}$$

where A_1, A_2, B_1, B_2, B_3, and B_4 are six integration constants. Let us consider a special case of choosing $A_1 = B_1 = B_2 = B_3 = 1$, $A_2 = 4$, $B_4 = 0$. Then we have

$$\begin{cases} p_1 = e^{-\frac{1}{3}t + \frac{1}{2}e^{-t-1} - 2e^{t-1}}, \\ p_2 = e^{-\frac{4}{3}t + \frac{1}{2}e^{-t-1} - 2e^{t-1}}, \\ r_1 = e^{\frac{1}{3}t - \frac{1}{2}e^{-t-1} + 2e^{t-1}}, \\ r_2 = 4e^{\frac{4}{3}t - \frac{1}{2}e^{-t-1} + 2e^{t-1}}, \\ q_1 = \frac{1}{2}e^{-t-1} + 2e^{t-1}, \\ q_2 = q_1 - 1. \end{cases} \tag{48}$$

Figure 5 shows the dynamics of this two-peakon for the potentials $u(x,t)$ and $v(x,t)$ determined by (48).

Remark 1. Although equations (33) and (38) share the same bi-Hamiltonian operators with different Hamiltonian functions, their peakon dynamics are very different. In the single-peakon

case, the peakon solution of (33) is in the type of traveling wave (see (37)), while the peakon solution of (38) is not, since the peak point does not change along with the time t (see (41)). In the two-peakon case, the collision of the two-peakon of equation (33) is discussed detailedly in [26], while the two-peakon of equation (38) never collide since their positions are satisfied with $q_2(t) = q_1(t) - B_1$ where B_1 is a nonzero constant.

Example 3.4. The two-component integrable system proposed by Song, Qu, and Qiao [25]

Choosing $H = \frac{1}{2}(uv - u_x v_x + u v_x - u_x v)$ casts equation (19) into

$$\begin{cases} m_t = \frac{1}{2}\left(m\left(uv - u_x v_x + u v_x - u_x v\right)\right)_x, \\ n_t = \frac{1}{2}\left(n\left(uv - u_x v_x + u v_x - u_x v\right)\right)_x, \\ m = u - u_{xx}, \quad n = v - v_{xx}, \end{cases} \tag{49}$$

which is exactly the equation derived by Song, Qu, and Qiao [25] from the viewpoint of geometry. This system possesses a Bi-Hamiltonian structure:

$$(m_t,\ n_t)^T = J\left(\frac{\delta H_2}{\delta m},\ \frac{\delta H_2}{\delta n}\right)^T = K\left(\frac{\delta H_1}{\delta m},\ \frac{\delta H_1}{\delta n}\right)^T, \tag{50}$$

where

$$J = \begin{pmatrix} 0 & \partial^2 + \partial \\ -\partial^2 + \partial & 0 \end{pmatrix}, \quad K = \begin{pmatrix} \partial m \partial^{-1} m \partial & \partial m \partial^{-1} n \partial \\ \partial n \partial^{-1} m \partial & \partial n \partial^{-1} n \partial \end{pmatrix}. \tag{51}$$

$$H_1 = \frac{1}{2}\int_{-\infty}^{+\infty} (u - u_x) n dx, \qquad H_2 = \frac{1}{4}\int_{-\infty}^{+\infty} (u - u_x)^2 (v + v_x) n dx. \tag{52}$$

Here in our paper, we want to derive the peakon solutions to this system. It is easy for one to check that the one-peakon solution of (49) takes the same form as (37). In general, by direct calculations, we can obtain the N-peakon dynamical system of (49) as follows:

$$\begin{cases} p_{j,t} = 0, \\ r_{j,t} = 0, \\ q_{j,t} = \frac{1}{6} p_j r_j + \frac{1}{2}\sum_{i,k=1}^{N} p_i r_k \left(sgn(q_j - q_i) sgn(q_j - q_k) - sgn(q_j - q_i) + sgn(q_j - q_k) - 1\right) e^{-|q_j - q_i| - |q_j - q_k|}. \end{cases} \tag{53}$$

If $N = 2$, then the two-peakon system reads as

$$\begin{cases} p_{1,t} = p_{2,t} = r_{1,t} = r_{2,t} = 0, \\ q_{1,t} = -\frac{1}{3} p_1 r_1 + \frac{1}{2}\left[p_1 r_2 \left(sgn(q_1 - q_2) - 1\right) - p_2 r_1 \left(sgn(q_1 - q_2) + 1\right)\right] e^{-|q_1 - q_2|}, \\ q_{2,t} = -\frac{1}{3} p_2 r_2 + \frac{1}{2}\left[p_1 r_2 \left(sgn(q_1 - q_2) - 1\right) - p_2 r_1 \left(sgn(q_1 - q_2) + 1\right)\right] e^{-|q_1 - q_2|}. \end{cases} \tag{54}$$

From the first equation of (54), we know

$$p_1 = A_1, \quad p_2 = A_2, \quad r_1 = B_1, \quad r_2 = B_2, \tag{55}$$

where A_1, A_2, B_1, and B_2 are four integration constants. If $A_1B_1 = A_2B_2$, then we have

$$\begin{cases} q_1(t) = \left\{-\frac{1}{3}A_1B_1 + \frac{1}{2}\left[A_1B_2\left(sgn(C_1)-1\right) - A_2B_1\left(sgn(C_1)+1\right)\right]e^{-|C_1|}\right\}t + \frac{C_1}{2}, \\ q_2(t) = q_1(t) - C_1. \end{cases} \tag{56}$$

If $A_1B_1 \neq A_2B_2$, then we arrive at:

$$\begin{cases} q_1(t) = -\frac{1}{3}A_1B_1t + \Gamma(t), \\ q_2(t) = -\frac{1}{3}A_2B_2t + +\Gamma(t), \end{cases} \tag{57}$$

where

$$\Gamma(t) = \frac{3(A_1B_2 + A_2B_1)}{2|A_1B_1 - A_2B_2|}sgn(t)\left(e^{-\frac{1}{3}|(A_1B_1-A_2B_2)t|} - 1\right) + \frac{3(A_1B_2 - A_2B_1)}{2(A_1B_1 - A_2B_2)}e^{-\frac{1}{3}|(A_1B_1-A_2B_2)t|}. \tag{58}$$

In particular, taking $A_1 = B_1 = 1$, $A_2 = 2$, and $B_2 = 5$ sends the two-peakon solution to the following form

$$\begin{cases} u(x,t) = e^{-|x-q_1(t)|} + 2e^{-|x-q_2(t)|}, \\ v(x,t) = e^{-|x-q_1(t)|} + 5e^{-|x-q_2(t)|}, \end{cases} \tag{59}$$

where

$$\begin{cases} q_1(t) = -\dfrac{t}{3} + \dfrac{7}{6}sgn(t)\left(e^{-3|t|} - 1\right) - \dfrac{1}{2}e^{-3|t|}, \\ q_2(t) = -\dfrac{10t}{3} + \dfrac{7}{6}sgn(t)\left(e^{-3|t|} - 1\right) - \dfrac{1}{2}e^{-3|t|}. \end{cases} \tag{60}$$

For the potential $u(x,t)$, the two-peakon collides at the moment $t = 0$, since $q_1(0) = q_2(0) = 0$. For $t < 0$, the tall and fast peakon with the amplitude 2 and peak position q_2 chases after the short and slow peakon with the amplitude 1 and peak position q_1. At the moment of $t = 0$, the two-peakon overlaps. After the collision ($t > 0$), the two-peakon separates, and the tall and fast peakon surpasses the short and slow one. Similarly, we may discuss the collision of the two-peakon for the potential $v(x,t)$. See Figures 6 and 7 for the two-peakon dynamics of the potentials $u(x,t)$ and $v(x,t)$.

4 One-component reduction under $v = 0$

If $v = 0$ and $G = 0$, (6) is reduced to the following one-component equation

$$m_t = F + F_x, \quad m = u - u_{xx}, \tag{61}$$

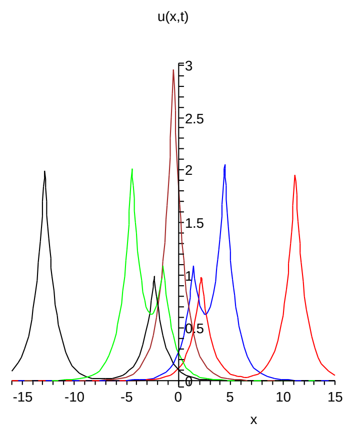

Figure 6: The two-peakon solution for the potential $u(x,t)$ given by (59). Red line: $t=-3$; Blue line: $t=-1$; Brown line: $t=0$ (collision); Green line: $t=1$; Black line: $t=3.5$.

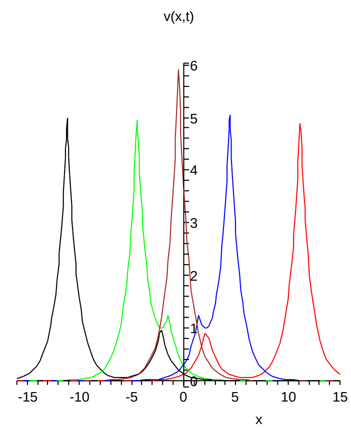

Figure 7: The two-peakon solution for the potential $v(x,t)$ given by (59). Red line: $t=-3$; Blue line: $t=-1$; Brown line: $t=0$ (collision); Green line: $t=1$; Black line: $t=3$.

with the Lax pair

$$U=\frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ 0 & 1 \end{pmatrix}, \qquad V=-\frac{1}{2}\begin{pmatrix} \lambda^{-2} & -\lambda^{-1}(u-u_x)-\lambda F \\ 0 & -\lambda^{-2} \end{pmatrix}, \tag{62}$$

where F is an arbitrary function of u, v and their derivatives. Let us discuss some special cases in the following examples.

Example 4.1. A linear model with N-peakon, N-kink, bell-shape, and weak hat-shape soliton solutions

Taking $F=u_x-u_{xx}$ sends equation (61) to:

$$m_t=m_x, \quad m=u-u_{xx}. \tag{63}$$

Suppose the single-peakon solution has the following form

$$u=p_1(t)e^{-|x-q_1(t)|}. \tag{64}$$

Substituting (64) into (63) and integrating against test function, we obtain the single-peakon of (63) as follows

$$u=A_1e^{-|x+t-B_1|}, \tag{65}$$

where A_1 and B_1 are two integration constants. Because (63) is a linear equation, we easily construct the N-peakon solution

$$u(x,t)=\sum_{j=1}^{N}A_je^{-|x+t-B_j|}, \tag{66}$$

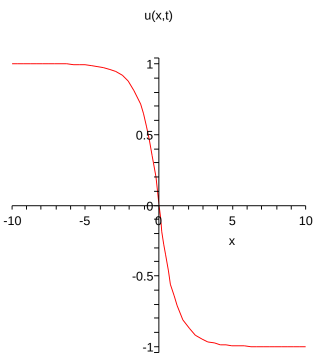

Figure 8: The kink solution (67) with $A_1 = 1$ and $B_1 = 0$ at the moment of $t = 0$.

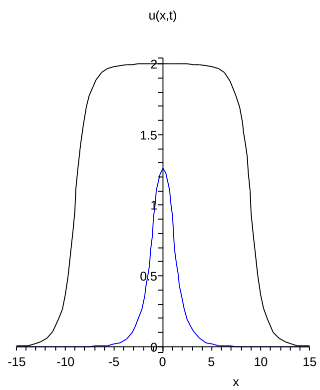

Figure 9: The solution (69) with $-A_1 = A_2 = 1$ at $t = 0$. Blue line (bell-shape soliton): $-B_1 = B_2 = 1$; Black line (hat-shape soliton): $-B_1 = B_2 = 9$.

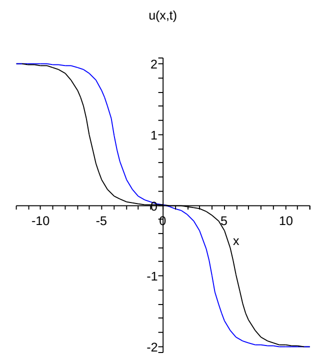

Figure 10: The two-kink solution determined by (69) with $A_1 = A_2 = 1$ at $t = 0$. Blue line: $-B_1 = B_2 = 4$; Black line: $-B_1 = B_2 = 6$.

where A_j and B_j are arbitrary integration constants.

Moreover, we may directly check that (63) possesses the following single weak-kink solution

$$u = A_1 sgn(x + t - B_1)\left(e^{-|x+t-B_1|} - 1\right), \tag{67}$$

and N-weak-kink solution

$$u(x,t) = \sum_{j=1}^{N} A_j sgn(x + t - B_j)\left(e^{-|x+t-B_j|} - 1\right), \tag{68}$$

where A_j and B_j are arbitrary constants. See Figure 8 for the profile of the single weak-kink solution. For $N = 2$, (68) reads as

$$u(x,t) = A_1 sgn(x + t - B_1)\left(e^{-|x+t-B_1|} - 1\right) + A_2 sgn(x + t - B_2)\left(e^{-|x+t-B_2|} - 1\right). \tag{69}$$

The interesting case is: when $A_1 = -A_2$ the two-weak-kink solution (69) turns into the soliton shape. Apparently, if the value $|B_1 - B_2|$ is small, it presents the usual bell-type soliton, but if the value $|B_1 - B_2|$ is a little large, the two-weak-kink solution takes on the “hat” shape. See Figure 9 for this bell shape and hat shape solutions with $B_2 = -B_1 = 1$ and $B_2 = -B_1 = 9$. When $A_1 = A_2$, (69) stands for the two-weak-kink solution, and see Figure 10 for its profile.

Example 4.2. A new integrable model with rational peakon solution

Let us select $F = mu$, then equation (61) reads as

$$m_t = mu + (mu)_x, \quad m = u - u_{xx}. \tag{70}$$

Assuming the single-peakon solution as the form (64), we arrive at

$$p_1(t) = -\frac{1}{t - A_1}, \qquad q_1(t) = \ln|t - A_1| + B_1, \tag{71}$$

where A_1 and B_1 are two arbitrary constants. Taking $A_1 = B_1 = 0$, then we have the single-peakon solution in the form of

$$u = -\frac{1}{t} e^{-|x - \ln|t||}. \tag{72}$$

As $t \to 0^-$ (t approaching 0 and $t < 0$), $p_1(t) = -\frac{1}{t} \to +\infty$, which means that the amplitude suddenly changes highly near $t = 0$. After the large crest, it is followed up immediately by deep troughs, since $p_1(t)$ is an odd function. This phenomenon looks very like the so-called rogue wave or monster wave [27, 28].

Example 4.3. A new model with periodic-peakon solution

Choosing $F = m(u + \frac{1}{u})$ yields the equation

$$m_t = m(u + \frac{1}{u}) + (m(u + \frac{1}{u}))_x, \quad m = u - u_{xx}. \tag{73}$$

Once again, suppose the single-peakon as the form (64), then we have

$$\begin{cases} p_{1,t} = p_1^2 + 1, \\ q_{1,t} = -\dfrac{p_{1,t}}{p_1}, \end{cases} \tag{74}$$

which admit the following solution

$$\begin{cases} p_1(t) = \tan(t + A_1), \\ q_1(t) = -\ln \mid \tan(t + A_1) \mid + B_1, \end{cases} \tag{75}$$

where A_1 and B_1 are two integration constants. Thus, we obtain the following one-peakon solution

$$u(x,t) \quad = \tan(t + A_1) e^{-|x + \ln|\tan(t + A_1)| - B_1|}, \tag{76}$$

which is actually a periodic-peakon. We will further study the multi-periodic-peakon solutions elsewhere.

We may generate many integrable peakon systems as per different choices of F and G in our model (6). So, our model provides a large class of peakon systems and covers almost all existing integrable peakon equations.

ACKNOWLEDGMENTS

The authors Xia and Zhou were supported by the National Natural Science Foundation of China (Grant No. 11271168). The author Qiao was supported by the Texas Norman Hackerman Advanced Research Program (Grant No. 003599-0001-2009).

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